

A new study for global dynamics and numerical simulation of a discrete-time computer virus propagation model

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Abstract— This work is devoted to conducting a new study for global dynamics and numerical simulation of a discrete-time computer virus propagation model, which was constructed in our recent work. By utilizing well-known results on asymptotic stability of discrete-time dynamical systems, we establish the global asymptotic stability of a unique viral equilibrium point, whereas only its local asymptotic stability was previously analyzed. After that, we investigate convergence and provide an error bound for the discrete-time model. Next, the step doubling strategy is applied to control errors. The result is that the accuracy of approximations generated by the discrete-time model is enhanced. The obtained results not only improve the ones constructed in the benchmark work, but also can be useful to study reliable numerical methods for mathematical models of malware. Finally, we present two numerical experiments that support and illustrate the theoretical findings of this study.

Tóm tắt— Bài báo đề xuất một nghiên cứu mới về động lực toàn cục và lời giải xấp xỉ cho một mô hình lan truyền virus máy tính với thời gian rời rạc. Bằng cách áp dụng các kết quả về sự ổn định tiệm cận của các hệ động lực rời rạc, nhóm tác giả thiết lập được tính chất ổn định tiệm cận toàn cục của điểm cân bằng dương duy nhất trong khi chỉ có tính chất ổn định tiệm cận địa phương của nó được thiết lập trước đó. Sau đó, nhóm tác giả phân tích sự hội tụ và đưa ra ước lượng sai số cho mô hình rời rạc. Tiếp theo, chiến lược gấp đôi bước lưới được áp dụng để kiểm soát sai số. Kết quả là độ chính xác của các lời giải xấp xỉ được nâng cao. Các kết quả thu được không những cải thiện nghiên cứu được đề xuất trước đây mà còn có thể mở rộng để xây dựng các phương pháp hiệu quả giải xấp xỉ các mô hình

toán học mô tả sự lan truyền của các phần mềm độc hại. Cuối cùng, nhóm tác giả thực hiện các thử nghiệm số để minh họa và hỗ trợ cho các kết quả lý thuyết.

Keywords — Global dynamics; numerical simulation; Lyapunov stability theory; computer viruses; NSFD methods; step doubling.

Từ khóa — Động lực toàn cục; mô phỏng số; lý thuyết ổn định Lyapunov; virus máy tính; phương pháp sai phân phi tiêu chuẩn; chiến lược gấp đôi bước lưới.

I. INTRODUCTION

We start by considering a mathematical model of the spreading of computer viruses proposed in [6] by Gan et al:

$$\dot{S} = \gamma_2 I + \eta_2 E - \mu S - \beta SI - \gamma_1 S,$$

$$\dot{I} = \beta SI - \mu I - \gamma_1 I - \gamma_2 I + \eta_1 E, \quad (1)$$

$$\dot{E} = \delta + \gamma_1 S + \gamma_1 I - \mu E - \eta_1 E - \eta_2 E,$$

where $S(t)$, $I(t)$ and $E(t)$ are functions of time t , which denote the average numbers of susceptible, infected and external computers, respectively; all the parameters take positive values due to biological reasons. More details of the model (1) were presented in [6]. In [17], Pham and Hoang developed the Mickens' methodology [14, 15, 16] to construct a nonstandard finite difference (NSFD) method for the continuous-time model (1), which has the following form:

$$\frac{S_{n+1} - S_n}{\varphi(\Delta t)} = \gamma_2 I_n + \eta_2 E_n - \mu S_{n+1}$$

$$- \beta S_{n+1} I_n - \gamma_1 S_{n+1},$$

$$\frac{I_{n+1} - I_n}{\varphi(\Delta t)} = \beta S_{n+1} I_n - \mu I_{n+1} - \gamma_1 I_{n+1}$$

$$- \gamma_2 I_n + \eta_1 E_n,$$

$$\frac{E_{n+1} - E_n}{\varphi(\Delta t)} = \delta + \gamma_1 S_{n+1} + \gamma_1 I_{n+1} - \mu E_{n+1}$$

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$$-\eta_1 E_n - \eta_2 E_n. \quad (2)$$

In the system (2):

- $\Delta t > 0$ is the step size.
- $(S_n, I_n, E_n)^T$ ($n = 1, 2, \dots, M$) are the intended approximations for $(S(t_n), I(t_n), E(t_n))^T$ with $t_n = n\Delta t, n = 0, 1, \dots, M$, respectively.
- The denominator $\varphi(\Delta t)$ is a positive function and satisfies $\varphi(\Delta t) = \Delta t + \mathcal{O}(\Delta t^2)$ as $\Delta t \rightarrow 0$.

By rigorous mathematical analysis, it has been shown that the NSFD scheme (2) correctly maintain essential mathematical properties of the continuous-time model (1), which include the positivity and boundedness of the solutions, the set of equilibria and the local asymptotic stability (LAS) of a unique viral equilibrium (VEP) for all $\Delta t > 0$. Also, the theoretical assertions were supported by a series of illustrative numerical simulations [17].

While the global asymptotic stability (GAS) of the continuous-time model has been established in [6], only the LAS of the discrete-time model (2) was proven in [17]; therefore, our first objective is to study the GAS of the VEP of (2). Here, it is important to remark that the analysis of global asymptotic stability (GAS) of dynamical systems is an important problem, which has numerous useful applications in both theory and practice contexts but it is generally not a straightforward task [5, 11]. In particular, the GAS problem of NSFD methods for dynamical systems governed by differential equations is very challenging. Recently, the GAS analysis of NSFD methods for some classes of mathematical models described by differential equations has been studied based on various approaches [3, 4, 8].

Motivated by the aforementioned reasons, we first establish the complete GAS of the NSFD model (2) by applying well-known results on asymptotic stability of discrete-time dynamical systems, which were proposed in [10]. With the help of these stability results and suitable Lyapunov functions, the GAS analysis of the NSFD method (2) is reduced to studying the

GAS of a one-dimensional dynamical system, in which the GAS problem is easily analyzed. As a result, the complete GAS of the unique VEP is established in Theorem 1, which improves the results presented in [17]. It is important to remark that the present approach has been used in [3, 8] to examine the GAS of computer virus propagation models.

In the second part of the present work, we study the convergence and give error estimates for the NSFD method (2). More clearly, we prove that it is only convergent of order one and give an estimate for its global error (Theorem 2). For this reason, the step doubling strategy [1, 2, 7] is applied to control errors. Consequently, the accuracy of approximations is enhanced. It should be emphasized that improving accuracy of NSFD methods for differential equations is very important and has attracted special attention of many researchers [9, 12, 13]. By using the step doubling strategy for the NSFD method (2), the local truncation error is controlled and the step size is automatically selected. The most important thing is that the original NSFD method (2) is dynamically consistent with the time-continuous model (1), so are the generated approximations.

In the third part of the present work, the theoretical insights are supported by two numerical examples, which provide evidence supporting the findings of this work.

This work is organized as follows: In Section II, we study the GAS of the NSFD model (2). Convergence and error estimates are investigated in Section III. A series of numerical experiments is reported in Section IV. Section V presents some concluding remarks and open problems.

II. ANALYSIS OF GLOBAL ASYMPTOTIC STABILITY

The aim of this section is to establish the GAS of the NSFD model (2). We recall that the explicit form of (2) is given by [17]:

$$S_{n+1} = \frac{S_n + \varphi(\Delta t)\gamma_2 I_n + \varphi(\Delta t)\eta_2 E_n}{1 + (\mu + \beta I_n + \gamma_1)\varphi(\Delta t)},$$

$$I_{n+1} = \frac{(1 - \varphi(\Delta t)\gamma_2)I_n + \varphi(\Delta t)\beta S_{n+1}I_n + \varphi(\Delta t)\eta_1 E_n}{1 + (\mu + \gamma_1)\varphi(\Delta t)},$$

$$E_{n+1} = \frac{\varphi(\Delta t)\delta + \varphi(\Delta t)\gamma_1(S_{n+1} + I_{n+1})}{1 + \mu\varphi(\Delta t)} + \frac{[1 - \varphi(\Delta t)(\eta_1 + \eta_2)]E_n}{1 + \mu\varphi(\Delta t)} \quad (3)$$

From now on, we always assume that the following assumption holds for $\varphi(\Delta t)$:

$$\varphi(\Delta t) < \left\{ \frac{1}{\gamma_2}, \frac{1}{\eta_1 + \eta_2} \right\}, \forall \Delta t > 0. \quad (4)$$

This condition implies that the NSFD method (2) preserves the positivity, boundedness and LAS of the continuous-time model (1) for all finite step sizes $\Delta t > 0$ [17]. We recall that the model (2) always possesses an unique VEP $E_V^* = (S^*, I^*, E^*)$ given by:

$$\begin{aligned} E^* &= \frac{\delta(\mu + \gamma_1)}{\mu(\mu + \gamma_1 + \eta_1 + \eta_2)}, \\ I^* &= \frac{\beta w - (\mu + \gamma_1 + \gamma_2)}{2\beta} \\ &+ \frac{\sqrt{[\beta w - (\mu + \gamma_1 + \gamma_2)]^2 + 4\beta\eta_1 E^*}}{2\beta} \quad (5) \\ S^* &= \frac{\delta}{\mu} - E^* - I^*, \end{aligned}$$

Where:

$$w = \frac{\delta(\eta_1 + \eta_2)}{\mu(\mu + \gamma_1 + \eta_1 + \eta_2)}.$$

Our main objective is to show the GAS of the VEP E_V^* . Let us denote $N_n := S_n + E_n + I_n$ for $n \geq 0$. Then, we represent the system (2) in a new form with the appearance of N_n :

$$\begin{aligned} \frac{N_{n+1} - N_n}{\varphi(\Delta t)} &= \delta - \mu N_{n+1}, \\ \frac{E_{n+1} - E_n}{\varphi(\Delta t)} &= \delta + \gamma_1(N_{n+1} - E_{n+1}) - \mu E_{n+1} \\ &\quad - \eta_1 E_n - \eta_2 E_n, \\ \frac{I_{n+1} - I_n}{\varphi(\Delta t)} &= \beta(N_{n+1} - E_{n+1} - I_{n+1})I_n - \mu I_{n+1} \\ &\quad - \gamma_1 I_{n+1} - \gamma_2 I_n + \eta_1 E_n. \quad (6) \end{aligned}$$

Note that the unique equilibrium E_V^* is now transformed to:

$$E_{new}^* = (N^*, E^*, I^*) = \left(\frac{\delta}{\mu}, E^*, I^* \right).$$

The following theorem shows the GAS of the unique equilibrium point of the discrete-time model (2) while only its LAS was shown in [17] (Theorem 2 in [17]).

Theorem 1. Assume that $\varphi(\Delta t)$ is a function with the property that

$$\varphi(\Delta t) < \frac{2I^*}{\eta_1 E^*}, \forall \Delta t > 0. \quad (7)$$

Then, the viral equilibrium point E_{new}^* of the model (6) is globally asymptotically stable with respect to the set $\mathbb{R}_+^3 := \{(N, E, I) | N, E, I \geq 0\}$.

Proof: The proof consists of three sequential steps outlined as follows.

Step 1: Reduce the 3-D system (6) to a 2-D system. Consider a Lyapunov candidate function defined by:

$$V_1(N_n, I_n, E_n) = (N_n - N^*)^2 = \left(N_n - \frac{\delta}{\mu} \right)^2.$$

It follows from the first equation of (6) that:

$$\begin{aligned} \Delta V_1(N_n, E_n, I_n) &:= V_1(N_{n+1}, I_{n+1}, E_{n+1}) \\ &\quad - V_1(N_n, I_n, E_n) \\ &= (N_{n+1} - N^*)^2 - (N_n - N^*)^2 \\ &= (N_{n+1} + N_n - 2N^*)(N_{n+1} \\ &\quad - N_n) \\ &= -\phi\mu(2 + \phi\mu)(N_{n+1} - N^*)^2. \end{aligned}$$

This implies that $\Delta V_1 \leq 0$ for all $(N_n, E_n, I_n) \in \mathbb{R}_+^3$ and $\Delta V_1 = 0$ if and only if $N_n = N^*$. Consequently, if we denote by G_1^* the largest positively invariant set contained in $G_1 := \{(N, E, I) \in \mathbb{R}_+^3 | \Delta V_1(N, E, I) = 0\}$, then

$$G_1^* := \{(N, E, I) \in \mathbb{R}_+^3 | N = N^*\}.$$

By using Theorem 3.2 in [10], it is sufficient to show that E_{new}^* is G_1^* -globally asymptotically stable. Since G_1^* is also a positively invariant set of (6), we only need to consider the following subsystem

$$\begin{aligned} \frac{E_{n+1} - E_n}{\varphi(\Delta t)} &= \delta + \gamma_1 \left(\frac{\delta}{\mu} - E_{n+1} \right) - \mu E_{n+1} \\ &\quad - \eta_1 E_n - \eta_2 E_n, \\ \frac{I_{n+1} - I_n}{\varphi(\Delta t)} &= \beta \left(\frac{\delta}{\mu} - E_{n+1} - I_{n+1} \right) I_n - \mu I_{n+1} \\ &\quad - \gamma_1 I_{n+1} - \gamma_2 I_n + \eta_1 E_n \quad (8) \end{aligned}$$

on a feasible set, which is given by $\Omega := \{(E, I) \in \mathbb{R}_+^2 | I + E \leq \frac{\delta}{\mu}\}$.

Step 2: Reduce the 2-D system (8) to a 1-D system. Consider a Lyapunov function candidate given by:

$$V_2(E_n, I_n) = (E_n - E^*)^2.$$

Then, we deduce from the first equation of the system (8) that:

$$\Delta V_2(E_n, I_n) := V_2(E_{n+1}, I_{n+1}) - V_2(E_n, I_n) = (E_{n+1} - E^*)^2 - (E_n - E^*)^2 = (E_{n+1} + E_n - 2E^*)(E_{n+1} - E_n) = -A(2 - A)(E_n - E^*)^2,$$

Where:

$$A := \frac{\varphi(\mu + \eta_1 + \eta_2 + \gamma_1)}{1 + \varphi(\gamma_1 + \mu)} \left[2 - \frac{\varphi(\mu + \eta_1 + \eta_2 + \gamma_1)}{1 + \varphi(\gamma_1 + \mu)} \right].$$

Hence, $\Delta V_2 \leq 0$ for all $(E_n, I_n) \in \mathbb{R}_+^2$ and $\Delta V_2 = 0$ if and only if $E_n = E^*$. Consequently, the largest positively invariant set contained in $G_2 := \{(E, I) \in \mathbb{R}_+^2 | \Delta V_2(E, I) = 0\}$ is defined by:

$$G_2^* := \{(E, I) \in \mathbb{R}_+^2 | E = E^*\}.$$

By using Theorem 3.2 in [10], we only need to prove that (E^*, I^*) is G_2^* -globally asymptotically stable. Because G_2^* is also a positively invariant set of (8), it is sufficient to analyze the following equation:

$$\frac{I_{n+1} - I_n}{\varphi(\Delta t)} = \beta \left(\frac{\delta}{\mu} - E^* - I_{n+1} \right) I_n - \mu I_{n+1} - \gamma_1 I_{n+1} - \gamma_2 I_n + \eta_1 E^* \quad (9)$$

on a feasible set $\Omega^* := \{I \in \mathbb{R}_+ | I \leq \frac{\delta}{\mu}\}$.

Step 3: Establish the GAS of the equation (9).

We rewrite the equation (9) in the form:

$$I_{n+1} = I_n - \frac{\varphi\beta}{1 + \varphi\beta I_n + \varphi(\mu + \gamma_1)} (I_n - I^-)(I_n - I^+),$$

where $I^+ = I^* > 0$ and $I^- < 0$ are two solutions of the quadratic equation:

$$I^2 - \frac{1}{\beta} \left[\beta \left(\frac{\delta}{\mu} - E^* - \gamma_2 \right) - (\mu + \gamma_1) \right] I - \frac{1}{\beta} \eta_1 E^* = 0. \quad (10)$$

Consider a Lyapunov function candidate given by:

$$V_3(I_n) = (I_n - I^*)^2.$$

By using (10), it is easy to check that

$$\Delta V_3 = -A_1(2 - A_1)(I_n - I^-)(I_n - I^*)^2.$$

Where:

$$A_1 = \frac{\varphi\beta(I_n - I^-)}{1 + \varphi\beta I_n + \varphi(\mu + \gamma_1)}.$$

Note that we deduce from (10) that

$$I^- = -\frac{\eta_1 E^*}{\beta I^*},$$

which implies that if (7) holds, then $2 - A_1 > 0$. Hence, we conclude that $\Delta V_3 \geq 0$ for all $I_n \geq 0$ and $\Delta V_3 = 0$ if and only if $I_n = I^*$. Consequently, from the Lyapunov stability theory [5], we obtain the GAS of I^* .

Now, the GAS of E_{new}^* of the full model (6) is obtained by combining steps 1-3. The proof is complete.

III. CONVERGENCE AND ERROR ESTIMATES

In this section, we study convergence and give an error bound for the numerical method (2). Let $y(t) = (S(t), I(t), E(t))^T$ be the exact solution of (1)

$y_n = (S_n, I_n, E_n)^T$ be the approximate solution generated by (2), $f(y(t))$ and $g(y_n, \Delta t)$ be the right-hand sides of (1) and (3), respectively. Then, it can be easily verified that

$$g(y, 0) = y, \quad \frac{\partial g(y, 0)}{\partial \Delta t} = f(y) \quad (11)$$

Thanks to the boundedness of $y(t)$ and y_n , we define:

$$C := \sup_{t \geq 0} \|y(t)\|, \quad D := \sup_{n \geq 0} \|y_n\|,$$

$$C_2 := \sup_{t \geq 0} \|y''(t)\|,$$

$$\Omega_{CD} := \{y \| y\| \leq \max\{C, D\}\}.$$

For each $\Delta t^* > 0$, we denote:

$$\Omega_D := \{(y, \Delta t) \mid \|y\| \leq D, 0 \leq \Delta t \leq \Delta t^*\},$$

and

$$\tau_1 := \max_{y \in \Omega_{CD}} \left\| \frac{\partial f(y)}{\partial y} \right\|,$$

$$\tau_2 := \max_{(y, \Delta t) \in \Omega_D} \left\| \frac{\partial^2 g(y, \Delta t)}{\partial \Delta t^2} \right\| + C_2. \quad (12)$$

Theorem 2. (Convergence and global error). *The NSFD method (2) is convergent of order one. Moreover, the following estimate is satisfied for $n \geq 0$:*

$$\|y(t_n) - y_n\| \leq \frac{\tau_2 \Delta t}{\tau_1} (e^{\tau_1 t_n} - 1).$$

Proof: First, let us denote $e_n = y(t_n) - y_n$. By using (2), (11) and the Taylor's theorem, we obtain:

$$y(t_{n+1}) = y(t_n + \Delta t) = y(t_n) + \Delta t f(y(t_n)) + \frac{\Delta t^2}{2} y''(\xi_n),$$

$$y_{n+1} = g(y_n, \Delta t) = y_n + \Delta t f(y_n) + \frac{\Delta t^2}{2} \frac{\partial^2 g}{\partial \Delta t^2}(y_n, \psi_n),$$

$$f(y(t_n)) - f(y_n) = \frac{\partial f}{\partial y}(\theta_n)(y(t_n) - y_n),$$

where $\xi_n, \psi_n \in (t_n, t_n + \Delta t)$ and θ_n is a point between $y(t_n)$ and y_n . Therefore:

$$\begin{aligned} y(t_{n+1}) - y_{n+1} &= (y(t_n) - y_n) + \Delta t (f(y(t_n)) - f(y_n)) + \frac{\Delta t^2}{2} \left(y''(\xi_n) - \frac{\partial^2 g}{\partial \Delta t^2}(y_n, \psi_n) \right) \\ &= (y(t_n) - y_n) + \Delta t \frac{\partial f}{\partial y}(\theta_n)(y(t_n) - y_n) + \frac{\Delta t^2}{2} \left(y''(\xi_n) - \frac{\partial^2 g}{\partial \Delta t^2}(y_n, \psi_n) \right), \end{aligned}$$

which implies that

$$\|e_{n+1}\| \leq \|e_n\| + \Delta t \tau_1 \|e_n\| + \tau_2 \Delta t^2,$$

where τ_1 and τ_2 are given by (12). By some simple algebraic manipulations, we have that

$$\begin{aligned} \|e_{n+1}\| &\leq (1 + \Delta t \tau_1)^{n+1} \|e_0\| \\ &\quad + \tau_2 \Delta t^2 \sum_{j=0}^n (1 + \Delta t \tau_1)^j. \end{aligned}$$

Since $\|e_0\| = 0$, the above estimate is simplified to

$$\begin{aligned} \|e_{n+1}\| &\leq \tau_2 \Delta t^2 \sum_{j=0}^n (1 + \Delta t \tau_1)^j = \\ &\frac{\tau_2 \Delta t}{\tau_1} [(1 + \Delta t \tau_1)^{n+1} - 1]. \end{aligned}$$

By utilizing the inequality $z + 1 \leq e^z$ for $z \geq 0$, we conclude that

$$\begin{aligned} \|e_{n+1}\| &\leq \frac{\tau_2 \Delta t}{\tau_1} (e^{(n+1)\Delta t \tau_1} - 1) \\ &= \frac{\tau_2 \Delta t}{\tau_1} (e^{\tau_1 t_{n+1}} - 1). \end{aligned}$$

The proof is complete.

IV. NUMERICAL EXPERIMENTS

In this section, we conduct two numerical examples to support and illustrate the results presented in Sections II and III.

Example 1. (Global asymptotic stability of the NSFD method). Consider the continuous-time model (1) with the following set of parameters

TABLE 1. PARAMETERS USER IN NUMERICAL SIMULATION

β	δ	μ	η_1	η_2	γ_1	γ_2
0.01	0.8	0.01	0.25	0.5	0.6	0.8

In this case, the VEP is given by $E_V^* = (23.6026, 16.1325, 40.2649)$, which is globally asymptotically stable. We will use $\varphi(\Delta t) = 1 - e^{-\Delta t}$ for the NSFD method (2), which satisfies the conditions (4) and (7). Numerical approximations generated by the NSFD method (2) are depicted in Figures 1-3.

It is clear that the GAS of the unique VEP is shown. This supports the results constructed in Section II.

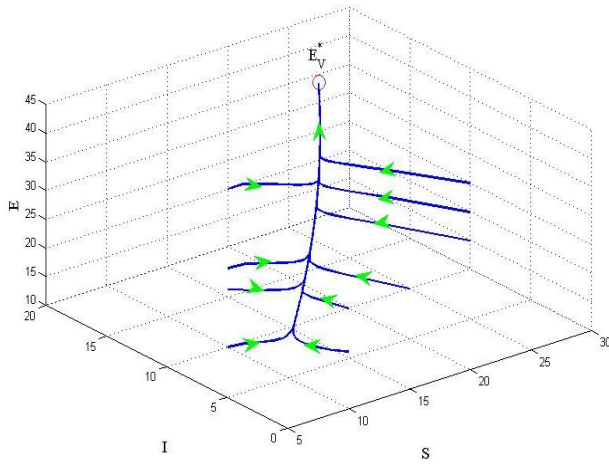


Figure 1. The phase spaces produced by applying the NSFD method with the step size $\Delta t = 2.0$

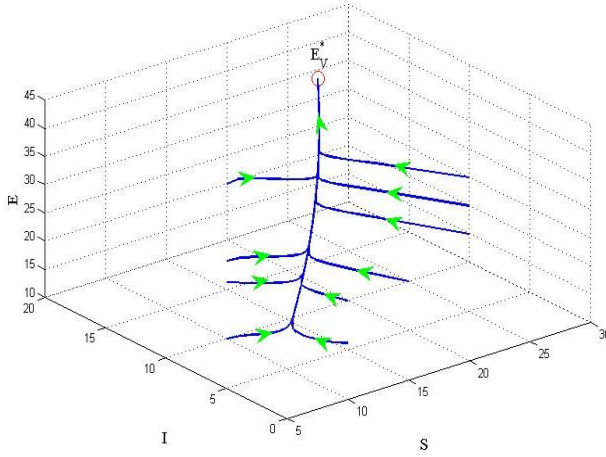


Figure 2. The phase spaces produced by using the NSFD method with the step size $\Delta t = 1.0$

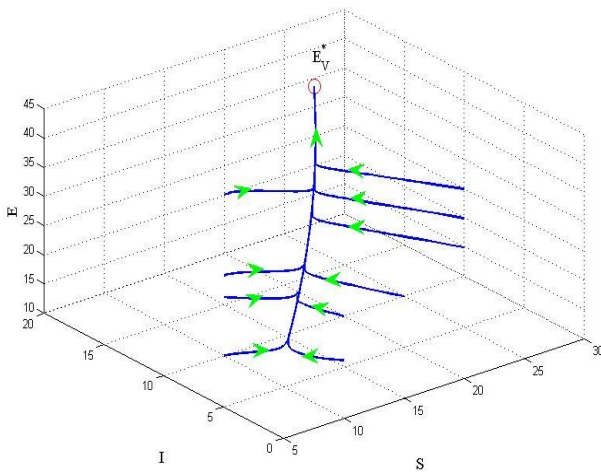


Figure 3. The phase spaces generated by employing the NSFD method with the step size $\Delta t = 0.1$

Example 2. (Error control based on step doubling). In this example, we will use the step doubling strategy to control errors of the NSFD method (2). Let us denote by y_n the solution using two steps of step size Δt starting from y_{n-2} and let $\tilde{y}_n$ be the solution taking one step of size $2\Delta t$ from y_{n-2} . Then, we have [1]

$$|y(t_n) - y_n| \approx \frac{|y_n - \tilde{y}_n|}{2}.$$

Consequently the local truncation error satisfies [1, 2, 7]:

$$|\tau_n(\Delta t)| = \left| \frac{y(t_n) - y_n}{\Delta t} \right| \approx r := \frac{|y_n - \tilde{y}_n|}{2\Delta t}.$$

Given an error tolerance ϵ . If $r \leq \epsilon$, we accept y_n as an approximation for $y(t_n)$ and continue to use the step size Δt in the next step. If $r > \epsilon$, then the current step is repeated with a new step size h_{new} given by [1, 2, 7]:

$$h_{new} = 0.9 \frac{\epsilon}{r}.$$

Next, let us consider the continuous-time model (1) on the interval $[0, 1]$ with the set of parameters in Table 1. We employ the NSFD method with the step doubling and record errors with respect to the Euclidean norm, which are computed at $t = 1$. Note that we admit the approximation generated by the classical four-stage explicit Runge-Kutta (RK4) method [1] using the step size of 10^{-6} as a benchmark/reference solution. The obtained results are reported in Table 2, where:

- N is the number of iterations used.
- Δt_{min} and Δt_{max} are the minimum and maximum step sizes selected, respectively.
- err is the error produced by applying the NSFD method with the step doubling.
- $err(\Delta t_{min}), err(\Delta t_{max}), err(\Delta t_n)$ are the errors corresponding to the step sizes $\Delta t_{min}, \Delta t_{max}$ and $\Delta t_n = 1/N$, respectively.

It is clear that the results shown in Table 2 is evidence supporting the theoretical assertions constructed in Sections II and III.

TABLE 2. NUMERICAL SIMULATION USING STEP DOUBLING

ϵ	N	Δt_{min}	Δt_{max}	err	$err(\Delta t_N)$
10^{-1}	28	0.0112	0.0451	0.0734	0.1320
10^{-2}	282	0.0027	0.0045	0.0068	0.0134
10^{-3}	2829	1.8645 e-004	4.5057 e-004	6.6939 -004	0.0013
10^{-4}	28297	2.0890 e-005	4.5057 e-005	6.6881 e-005	1.3375 e-004
10^{-5}	28298 0	1.2301 e-006	4.5057 e-006	6.6878 e-006	1.3375 e-005
10^{-6}	28272 22	5.9011 e-008	4.5057 e-007	6.7170 e-007	1.3352 e-006

V. CONCLUDING REMARKS AND DISCUSSIONS

We have conducted a new study for global dynamics and numerical simulation of a discrete-time computer virus propagation model proposed in [17]. By utilizing well-known results on asymptotic stability of discrete-time dynamical systems [10], we have established the global asymptotic stability of the unique viral equilibrium point while only its local asymptotic stability was analyzed in [17]. Moreover, we have investigated the convergence and given an error bound for the NSFD method (2). Then, the step doubling strategy has been applied to control errors. Consequently, the accuracy of approximations generated by the discrete-time model is enhanced. The obtained results not only improve the ones presented in [17] but also can be useful to study reliable numerical methods for mathematical models of computer viruses, malware and rumors. Finally, the theoretical insights have been illustrated through a series of numerical experiments.

Our next objectives are to extend the present approach and obtained results to study mathematical models arising in real-world situations. In particular, higher-order reliable numerical methods for systems modeling the spreading of computer viruses, malware and rumors will be of special interest.

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