Extrapolated nonstandard numerical schemes for solving an epidemiological model for computer viruses

Pham Hoai Thu, Hoang Manh Tuan

***Abstract*— The aim of this work is to construct high-order numerical schemes that preserve the dynamical properties of a mathematical model describing the spread of computer viruses on the Internet. For this purpose, we first apply Mickens' methodology to formulate a dynamically consistent nonstandard finite difference (NSFD) scheme for the model under consideration. After that, the constructed NSFD scheme is combined with Richardson's extrapolation method to generate higher-accuracy numerical approximations. The result is that we obtain extrapolated numerical schemes that not only preserve the dynamical properties of the computer virus propagation model but also provide higher-accuracy numerical approximations. In addition, a set of numerical examples is conducted to illustrate and support the theoretical findings and to show the advantages of the proposed numerical schemes.**

***Tóm tắt*— Trong bài báo này, các lược đồ sai phân phi tiêu chuẩn có cấp chính xác cao và bảo toàn các tính chất hệ động lực của một mô hình lan truyền virus máy tính trên Internet được xây dựng. Đầu tiên, nhóm tác giả áp dụng phương pháp luận của Mickens để đề xuất một lược đồ sai phân phi tiêu chuẩn tương thích động lực học với mô hình lan truyền virus máy tính. Sau đó, lược đồ sai phân được kết hợp với phương pháp ngoại suy Richardson để tạo ra các lược đồ sai phân có cấp chính xác cao. Kết quả là nhóm tác giả thu được các lược đồ sai phân phi tiêu chuẩn không những bảo toàn các tính chất hệ động lực của mô hình liên tục mà còn cung cấp các lời giải xấp xỉ có cấp chính xác cao. Cuối cùng, các mô phỏng số được thực hiện để minh họa và hỗ trợ cho các kết quả lý thuyết cũng như để chứng minh ưu thế của các lược đồ sai phân khác thường được xây dựng.**

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Keywords*— *NSFD schemes; dynamic consistency; computer viruses; high-order; Richardson's extrapolation.

Từ khóa*— *Lược đồ sai phân phi tiêu chuẩn; tương thích động lực; virus máy tính; cấp chính xác cao; ngoại suy Richardson.

I. INTRODUCTION

Mathematical modeling and analysis of the spread of computer viruses and malware can provide us with many useful applications in science and technology. This topic has strongly attracted the attention of mathematicians and engineers for many years (see, for instance, [1, 9, 28, 31, 33-35] and references therein). Recently, we have studied qualitative properties and reliable numerical schemes for some mathematical models of computer viruses and malware [5, 12-15].

In this work, we reconsider a mathematical model describing the spread of computer viruses on the Internet, which was proposed and analyzed by Yang and Yang in [34]. In this model, we call computers as nodes; a node is called internal or external depending on whether it is connected to the Internet or not; and all computers are classified into three groups depending on their status with respect to computer viruses, that are:

- Susceptible S: the group of uninfected computers.
- Latent L: the group of infected computers without any exploiting viral replication.
- Exploding B: the group of infected computers with exploding viral replication.

Let $S(t)$, $L(t)$ and $B(t)$ be the numbers of uninfected, latent, and exploding internal nodes at time t, respectively. Based on some technique hypotheses, the following nonlinear system of ordinary differential equations was formulated to describe the transmission dynamics of computer viruses on the Internet (see [34]):

$$\frac{dS(t)}{dt} = \mu_1 - [\beta_1(L(t)) + \beta_2(B(t)) + \theta]S(t) + \gamma_1 L(t) + \gamma_2 B(t) - \delta S(t),$$
$$\frac{dI(t)}{dt} = \mu_2 + [\beta_1(L(t)) + \beta_2(B(t)) + \theta]S(t) - (\alpha + \gamma_1 + \delta)L(t),$$
$$\frac{dB(t)}{dt} = \alpha L(t) - (\gamma_2 + \delta)B(t), \qquad (1)$$

where all the parameters are assumed to be positive; β_1 and β_2 are functions having continuous second derivatives with $\beta_1', \beta_2' > 0$ and $\beta_1'', \beta_2'' < 0$. Details of this model can be found in [34].

Since it is impossible to determine the exact solution, our objective is to construct reliable numerical schemes for solving model (1). For this purpose, we first apply Mickens' methodology [23-27] to propose a nonstandard finite difference (NSFD) scheme preserving the positivity, boundedness and asymptotic stability of the model (1). Although the main advantage of NSFD methods is that they have the ability to preserve essential properties of differential equation models regardless of chosen step sizes (see, for example, [23-27, 29, 30] and references therein), most of them are only convergent of order 1. Therefore, the problem of improving the accuracy of NSFD schemes has been studied by many researchers with various approaches (see, for instance, [4, 6, 7, 10, 11, 16, 20-22]). For this reason, after constructing the (first-order) dynamically consistent NSFD scheme, we combine it with Richardson's extrapolation method [18, 32] to generate higher-accuracy numerical approximations. The result is that we obtain extrapolated numerical schemes that not only preserve the positivity, boundedness and asymptotic stability of the model (1) but also have higher-accuracy. Note that the combination of NSFD schemes and Richardson's extrapolation technique has been used in [11] to improve the numerical solution of population models; however, this work only deals with the positivity and boundedness, not the asymptotic stability of the population models.

Along with the theoretical analysis, a set of numerical examples is performed to illustrate and support the theoretical findings and to show the advantages of the constructed numerical schemes. The obtained results show that there is a good agreement between the theoretical findings and the numerical results.

The plan of the paper is as follows: NSFD schemes are proposed and analyzed in Section II. Numerical examples are conducted in Section III. The last section provides some conclusions and discussions.

II. Construction of NSFD schemes

In this section, we will construct NSFD schemes for model (1). For this purpose, we now consider the model (1) on a finite time interval [0, T] and partition this interval by a uniform mesh

$$0 = t_0 < t_1 < \ldots < t_i < \ldots < t_{N-1} < t_N = T,$$

where $t_{n+1} - t_n = \Delta t$ for $i = 0, 1, \ldots, N-1$. Let us denote by $(S_n, L_n, B_n)^T (n = 1, 2, \ldots, N)$ the intended approximations for $(S(t_n), L(t_n), B(t_n))^T$, respectively. By applying the rules for the construction of NSFD schemes [23-27], model (1) is numerically approximated as follows:

$$\dot{S}(t_n) \approx \frac{S_{n+1} - S_n}{\phi(\Delta t)},$$
$$\dot{L}(t_n) \approx \frac{L_{n+1} - L_n}{\phi(\Delta t)},$$
$$\dot{B}(t_n) \approx \frac{B_{n+1} - B_n}{\phi(\Delta t)},$$

and

$$\mu_1 - [\beta_1(L(t_n)) + \beta_2(B(t_n)) + \theta]S(t_n) + \gamma_1 L(t_n) + \gamma_2 B(t_n) - \delta S(t_n)$$
$$\approx \mu_1 - [\beta_1(L_n) + \beta_2(B_n) + \theta]S_{n+1} + \gamma_1 L_n + \gamma_2 B_n - \delta S_{n+1},$$
$$\mu_2 + [\beta_1(L(t_n)) + \beta_2(B(t_n)) + \theta]S(t_n) - (\alpha + \gamma_1 + \delta)L(t_n)$$
$$\approx \mu_2 + [\beta_1(L_n) + \beta_2(B_n) + \theta]S_{n+1} - (\alpha + \delta)L_{n+1} + \gamma_1 L_n,$$
$$\alpha L(t_n) - (\gamma_2 + \delta)B(t_n)$$
$$\approx \alpha L_{n+1} - \gamma_2 B_n + \delta B_{n+1},$$

where $\phi(\Delta t) = \Delta t + O(\Delta t^2)$ as $\Delta t \to 0$ is called a nonstandard denominator function. For the sake of convenience, the variable Δt

will be omitted in some places. The above discretization process leads to the following NSFD scheme for model (1):

$$\frac{S_{n+1}-S_n}{\phi(\Delta t)} = \mu_1 - [\beta_1(L_n) + \beta_2(B_n) + \theta]S_{n+1} + \gamma_1 L_n + \gamma_2 B_n - \delta S_{n+1},$$
$$\frac{L_{n+1}-L_n}{\phi(\Delta t)} = \mu_2 + [\beta_1(L_n) + \beta_2(B_n) + \theta]S_{n+1} - (\alpha+\delta)L_{n+1} + \gamma_1 L_n,$$
$$\frac{B_{n+1}-B_n}{\phi(\Delta t)} = \alpha L_{n+1} - \gamma_2 B_n + \delta B_{n+1}. \quad (2)$$

In order to analyze dynamical properties of the NSFD model (2), we use the notation $N_n = S_n + L_n + B_n$ for $n \geq 0$ and introduce the following hypothesis

$$\phi(\Delta t) < min\left\{\frac{1}{\gamma_1}, \frac{1}{\gamma_2}\right\} \text{ for all } \Delta t > 0. \quad (3)$$

Theorem 1 (Positivity and boundedness). *Let $\phi(\Delta t)$ be a function satisfying the condition (3). Then, the following assertions hold for the NSFD scheme (2):*

1) The set $\mathbb{R}_3^+ = \{(S,L,B)|S,L,B \geq 0\}$ is a positively invariant set of the NSFD scheme (2), that is $S_n, L_n, B_n \geq 0$ for $n > 0$ whenever $S(0), L(0), B(0) \geq 0$.

2) The sequence $\{N_n\}_{n=0}^{\infty}$ monotonically converges to $N^* := \mu/\delta$ as $n \to \infty$, where $\mu := \mu_1 + \mu_2$.

Proof of Part 1. This part is proved by mathematical induction. Indeed, we first transform the system (3) to the explicit form as follows:

$$S_{n+1} = \frac{S_n + \phi(\mu_1 + \gamma_1 L_n + \gamma_2 B_n)}{1 + \phi(\beta_1(L_n) + \beta_2(L_n) + \theta + \delta)},$$
$$L_{n+1} = \frac{(1-\phi\gamma_1)L_n + \phi\gamma_2}{1+\phi(\alpha+\delta)} + \frac{\phi(\beta_1(L_n) + \beta_2(L_n) + \theta)S_{n+1}}{1+\phi(\alpha+\delta)}, \quad (4)$$
$$B_{n+1} = \frac{(1-\phi\gamma_2)B_n + \phi\alpha L_{n+1}}{1+\phi\delta}$$

It follows from (3) that $1 - \phi\gamma_1 > 0$ and $1 - \phi\gamma_2 > 0$. Hence, we deduce that $S_{n+1}, L_{n+1}, B_{n+1} \geq 0$ if $S_n, L_n, B_n \geq 0$. The proof for this part is complete.

Proof of Part 2. By adding side-by-side the first and second equations of the system (2), we have:

$$\frac{N_{n+1} - N_n}{\phi} = \mu - \delta N_{n+1},$$

which implies that

$$N_{n+1} = \left(\frac{\mu}{\delta} - N_0\right)\left(\frac{1}{1+\phi\delta}\right)^n + \frac{\mu}{\delta}.$$

therefore, the monotone convergence of the sequence $\{N_n\}$ is proved due to the fact that $[1/(1+\phi\delta)] \in (0,1)$. The proof for this part is complete.

Next, we consider the following system of algebraic equations to determine the equilibria of model (2):

$$S_{n+1} = S_n, \;\; I_{n+1} = I_n, \;\; E_{n+1} = E_n.$$

It is easy to verify that the equilibrium points of the discrete model (2) and the continuous model (1) are identical. Hence, the NSFD scheme also has a unique viral (positive) equilibrium point $E_* = (S_*, B_*, L_*)$.

We now apply Lyapunov's indirect method [8, 19] to investigate the asymptotic stability of model (2). Following this method, we need to determine conditions that guarantee that all the eigenvalues of the Jacobian matrix of system (2) evaluated at the unique viral equilibrium point lie strictly inside the unit circle. By applying the approach used in [17], we obtain the following theorem.

Theorem 2 (Asymptotic stability). *There exists a positive number ϕ_* that only depends on the parameters of the model (1) and plays as a stability threshold of the NSFD model (2), i.e., the unique viral equilibrium point is asymptotically stable whenever*

$$\phi(\Delta t) < \phi_* \quad \text{for all } \Delta t > 0. \quad (5)$$

Theorems 1 and 2 lead to the following result.

Theorem 3. *The NSFD scheme (2) is dynamically consistent with respect to the positivity, boundedness and asymptotic stability of the computer virus propagation model (1) provided that*

$$\phi(\Delta t) < \Delta t_* := min\left\{\frac{1}{\gamma_1}, \frac{1}{\gamma_1}, \phi_*\right\} \quad (6)$$

for all $\Delta t > 0$, where ϕ_* is defined in Theorem 2.

Remark 1. *1) It is easy to design an algorithm for computing the number Δt_* in the condition (6).*

2) There are many denominator functions satisfying the condition (6), a simple function is (see [23-27]).

$$\phi(\Delta t) = \frac{1 - e^{-\tau \Delta t}}{\tau}, \quad \tau > \frac{1}{\Delta t_*}.$$

Some general families of denominator functions satisfying (6) were introduced in [6].

Before constructing extrapolated NSFD schemes, we need to analyze the convergence and error estimates of the NSFD scheme (2).

Theorem 4. *The NSFD scheme (2) is convergent of order 1.*

Proof. First, we set:

$$y(t) = (S(t), L(t), B(t))^T, y_n = (S_n, L_n, B_n)^T,$$

and denote by $f(y(t))$ and $g(y_n, \Delta t)$ the right-side functions of models (1) and (4), respectively. It is easy to verify that

$$g(y, 0) = y, \quad \frac{\partial g(y,0)}{\partial \Delta t} = f(y).$$

from to the boundedness of $y(t)$ and $\{y_n\}$ it is valid to set

$$c := \sup_{t \geq 0} \| y(t) \|,$$
$$d := \sup_{t \geq 0} \| y_n \|,$$
$$c_2 := \sup_{t \geq 0} \| y''(t) \|,$$
$$\Omega_{CD} := \{ y \mid \| y \| \leq \max\{c, d\}\}.$$

for each arbitrary but fixed step size $\Delta t > 0$, we denote

$$\Omega_D := \{(y, \Delta t) \mid \| y \| \leq d, 0 < \Delta t < \Delta t^*\}$$

and

$$\tau_1 := \max_{y \in \Omega_{CD}} \left\| \frac{\partial f(y)}{\partial y} \right\|$$
$$\tau_2 := \max_{(y,\Delta t) \in \Omega_D} \left\| \frac{\partial^2 g(y)}{\partial y^2} \right\| + c_2. \qquad (7)$$

It follows from Taylor's theorem that

$$y(t_{n+1}) = y(t_n + \Delta t)$$
$$= y(t_n) + \Delta t y'(t_n) + \frac{\Delta t^2}{2} y''(\xi_n)$$
$$= y(t_n) + \Delta t f(y(t_n)) + \frac{\Delta t^2}{2} y''(\xi_n),$$

$$y_{n+1} = g(y_n, \Delta t)$$
$$= g(y_n, 0) + \Delta t \frac{\partial g}{\partial \Delta t}(y_n, 0) + \frac{\Delta t^2}{2} \frac{\partial^2 g}{\partial \Delta t^2}(y_n, \psi_n)$$

$$= y_n + \Delta t f(y_n) + \frac{\Delta t^2}{2} \frac{\partial^2 g}{\partial \Delta t^2}(y_n, \psi_n),$$

$$f(y(t_n)) - f(y_n) = \frac{\partial f}{\partial y}(\theta_n)(y(t_n) - y_n)$$

where ξ_n, $\psi_n \in (t_n, t_n + \Delta t)$ and $\theta_n \in (y(t_n), y_n)$. Therefore,

$$y(t_{n+1}) - y_{n+1} = (y(t_n) - y_n) + \Delta t(f(y(t_n) - f(y_n)) + \frac{\Delta t^2}{2}(y''(\xi_n) - \frac{\partial^2 g}{\partial \Delta t^2}(y_n, \psi_n)$$
$$= (y(t_n) - y_n) + \Delta t \frac{\partial f}{\partial y}(\theta_n)(y(t_n) - y_n) + \frac{\Delta t^2}{2}(y''(\xi_n) - \frac{\partial^2 g}{\partial \Delta t^2}(y_n, \psi_n)),$$

which implies that

$$\| e_{n+1} \| \leq \| e_n \| + \Delta t \tau_1 \| e_n \| + \tau_2 \Delta t^2$$
$$= (1 + \Delta t \tau_1) \| e_n \| + \tau_2 \Delta t^2,$$

where $e_n := y(t_n) - y_n$; τ_1 and τ_2 are given by (7). Hence,

$$\| e_{n+1} \|$$
$$\leq (1 + \Delta t \tau_1)[(1 + \Delta t \tau_1) \| e_{n-1} \| + \tau_2 \Delta t^2] + \tau_2 \Delta t^2$$
$$= (1 + \Delta t \tau_1)^2 \| e_{n-1} \| + \tau_2 \Delta t^2 (1 + \Delta t \tau_1) + \tau_2 \Delta t^2$$
$$\leq \ldots \leq (1 + \Delta t \tau_1)^{n+1} \| e_0 \| + \tau_2 \Delta t^2 \sum_{j=0}^{n} (1 + \Delta t \tau_1)^j.$$

By using $\| e_0 \| = 0$, we obtain:

$$\| e_{n+1} \| \leq \tau_2 \Delta t^2 \sum_{j=0}^{n} (1 + \Delta t \tau_1)^j$$
$$= \frac{\tau_2 \Delta t}{\tau_1}[(1 + \Delta t \tau_1)^{n+1} - 1].$$

By applying the well-known inequality $z + 1 \leq e^z$ for $z \geq 0$, the following estimate is obtained:

$$e_{n+1} \| \leq \frac{\tau_2 \Delta t}{\tau_1}(e^{(n+1)\Delta t \tau_1} - 1 = \frac{\tau_2 \Delta t}{\tau_1}(e^{\tau_1 t_{n+1}} - 1).$$

This is the desired conclusion. The proof is complete.

We now construct an extrapolated NSFD scheme for model (1) by combining the NSFD scheme (2) with Richardson's extrapolation method. It is well-known that Richardson's extrapolation method generates higher-accuracy results by combining lower formulas [18, 32]. This method has been widely used in the numerical analysis of differential equations and integrals [2, 3].

Let us denote by:

${y_n}^{\Delta t/2} := (S_n^{\Delta t/2}, L_n^{\Delta t/2}, B_n^{\Delta t/2})^T$ and

$y_n^{\Delta t} := (S_n^{\Delta t}, L_n^{\Delta t}, B_n^{\Delta t})^T (n \geq 1)$ the numerical solutions generated by the first-order NSFD scheme (2) using the step sizes $\Delta t/2$ and Δt, respectively. Note that $y_n^{\Delta t/2}$ and $y_n^{\Delta t}$ are O(Δt) approximations. Then,

$$z_n^{\Delta t} := 2y_n^{\Delta t/2} - y_n^{\Delta t} \qquad (8)$$

provides an O(Δt^2) approximation formula for model (1) (see [3]). By the similar way, we can obtain higher-accuracy numerical approximations. For example, third-order and fourth-order formulas can be obtained as follows:

$$v_n^{\Delta t} := \frac{4z_n^{\Delta t/2} - z_n^{\Delta t}}{3} \qquad (9)$$

and

$$w_n^{\Delta t} := \frac{8v_n^{\Delta t/2} - v_n^{\Delta t}}{7} \qquad (10)$$

respectively. In general, the $(n+1)-$ order formula is given by

$$Y^{\Delta t} := \frac{2^n X^{\Delta t/2} - X^{\Delta t}}{2^n - 1},$$

where $X^{\Delta t}$ is an $n-$order formula.

In numerical examples performed in the next section, we will observe the convergence and errors of the extrapolated NSFD schemes (8) - (10).

III. NUMERICAL EXAMPLES

In this section, we perform some numerical experiments to illustrate the theoretical results and show the advantages of the NSFD schemes.

Example 1 (Comparing the NSFD scheme (2) with standard schemes). *Consider model (1) with the data given by*

$$\beta_1(L) = \frac{0.1L}{1+L}, \quad \beta_2(L) = \frac{0.2L}{1+L}, \quad \gamma_1 = 0.5,$$
$$\gamma_2 = 0.25, \quad \mu_1 = 0.5, \quad \mu_2 = 0.25,$$
$$\alpha = 0.2, \quad \theta = 0.5, \quad \delta = 0.01,$$

In this case, the viral equilibrium point $E_* = (25.1019,\ 28.2033,\ 21.6948)$ *is asymptotically stable. Numerical approximations generated by the standard Euler scheme, the second-order RungeKutta (RK2) scheme and the NSFD scheme (2) (with* $\phi(h) = 1 - e^{-h}$ *) for* $(S(0), L(0), B(0)) = (25,\ 20,\ 16)$ *are depicted in Figures 1 and 2.*

It is clear that the Euler and RK2 schemes fail to preserve the asymptotic stability, meanwhile, the NSFD scheme (2) correctly preserves the dynamics of model (1). Furthermore, we observe from Figures 3 and 4 that the dynamics of the NSFD scheme (2) do not depend on the chosen step sizes.

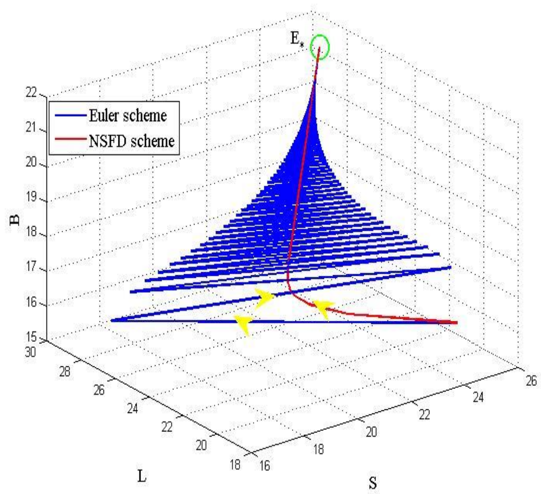

Figure 1. Phase spaces generated by the Euler scheme and the NSFD scheme (2) with $\Delta t = 1.45$ after 1000 iterations

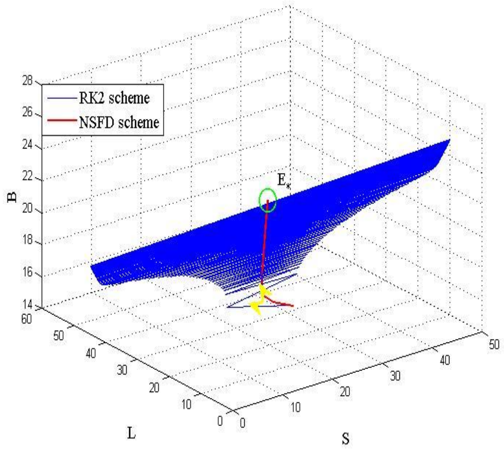

Figure 2. Phase spaces generated by the RK2 scheme and the NSFD scheme (2) with $\Delta t = 1.5$ after 1000 iterations

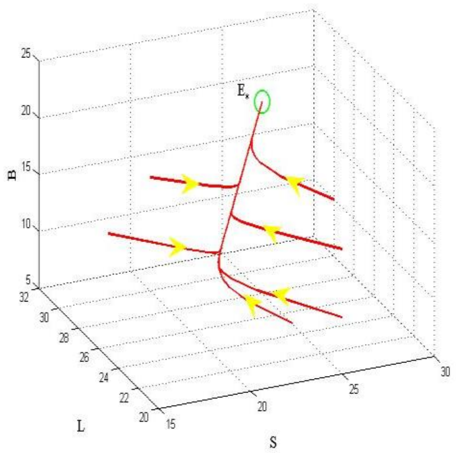

Figure 3. Phase spaces generated by the NSFD scheme (2) with some different initial data, $\Delta t = 2.0$ and $t \in [0,2000]$

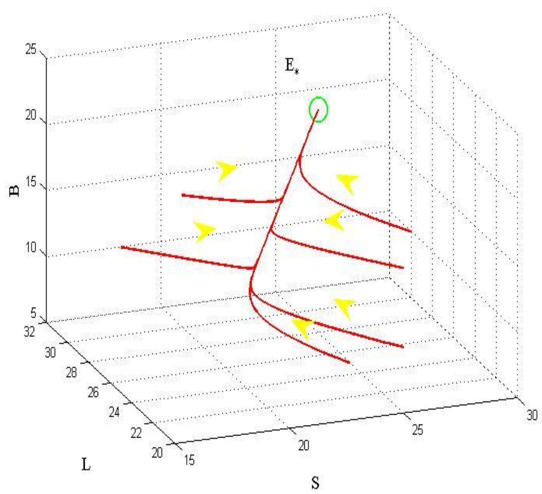

Figure 4. Phase spaces generated by the NSFD scheme (2) with some different initial data, $\Delta t = 0.01$ and $t \in [0,2000]$

Example 2 (Errors and convergence of the extrapolated NSFD schemes). *In this example, we analyze the convergence and errors of the extrapolated schemes (8) - (10). For this purpose, we consider model (1) with the following data*

$$\beta_1(L) = \frac{0.25L}{1+L}, \quad \beta_2(L) = \frac{0.5B}{1+B}, \quad \gamma_1 = 0.5,$$
$$\gamma_2 = 0.75, \quad \mu_1 = 0.5, \quad \mu_2 = 0.25,$$
$$\alpha = 0.25, \quad \theta = 0.55, \quad \delta = 0.01,$$

Since it is impossible to determine the exact solution, we admit the numerical solution generated by the fourth-order Runge-Kutta scheme using the step size $\Delta t = 10^{-5}$ as a reference (benchmark) solution.

Our main investigation is focused on approximations for model (1) at the end of the time interval, i.e., at $T = 100$. More clearly, we now use the NSFD schemes (2) and (8)-(10) with $\phi(\Delta t) = \frac{1-e^{-1.6\Delta t}}{1.6}$ to compute the numerical approximations at $T = 100$. The obtained results are reported in Table I-IV. In these tables, the errors of the schemes are defined by

$$error(\Delta t) = |S(t_N) - S_N| + |L(t_N) - L_N| + |B(t_N) - B_N|,$$

and

$$rate := log_{(\Delta t_1/\Delta t_2)} \left(error(\Delta t_1) / error(\Delta t_2)\right)$$

is used to approximate the rates of the convergence of the numerical schemes (see [2]).

TABLE 1. ERRORS AND CONVERGENCE RATES OF (2)

Δt	$error$	$rate$
0.5	1.8917	
0.25	0.9951	0.9268
0.2	0.8033	0.9593
0.1	0.4085	0.9757
0.05	0.2058	0.9889
0.025	0.1033	0.9947
0.02	0.0827	0.9968
0.01	0.0414	0.9980

TABLE 2. ERRORS AND CONVERGENCE RATES OF (8)

Δt	$error$	$rate$
0.5	0.0984	
0.25	0.0220	2.1608
0.2	0.0136	2.1450
0.1	0.0032	2.1130
0.05	7.5200E − 004	2.0682
0.025	1.8317E − 004	2.0376
0.02	1.1660E − 004	2.0243
0.01	$2.8829e - 005$	2.0159

TABLE 3. ERRORS AND CONVERGENCE RATES OF (9)

Δt	$error$	$rate$
0.5	0.0035	
0.25	$6.2628e - 004$	2.4662
0.2	$3.4258e - 004$	2.7036
0.1	$4.8549e - 005$	2.8189
0.05	$6.4410e - 006$	2.9141
0.025	$8.2896e - 007$	2.9579
0.02	$4.2656e - 007$	2.9775
0.01	$5.3879e - 008$	2.9849

TABLE 4. ERRORS AND CONVERGENCE RATES OF (10)

Δt	$error$	$rate$
0.5	$2.2137e - 004$	
0.25	$1.5636e - 005$	3.8235
0.2	$6.5443e - 006$	3.9034
0.1	$4.2557e - 007$	3.9428
0.05	$2.7242e - 008$	3.9655
0.025	$1.4186e - 009$	4.2633
0.02	$6.3965e - 010$	3.5694
0.01	$1.4966e - 010$	2.0956

It is clear that the accuracy of the NSFD scheme (2) is improved by the extrapolated NSFD schemes. It should be emphasized that the underlying NSFD scheme (2) is dynamically consistent with model (1).

To end this example, we observe errors over the time interval $[0,100]$ generated by the extrapolated NSFD scheme (9) with $\phi(\Delta t) = h$ $(ENSFD2)$ and the RK2 scheme using the step size $\Delta t = 10^{-3}$. From Figure 5, we see that the ENSFD2 is better than the RK2.

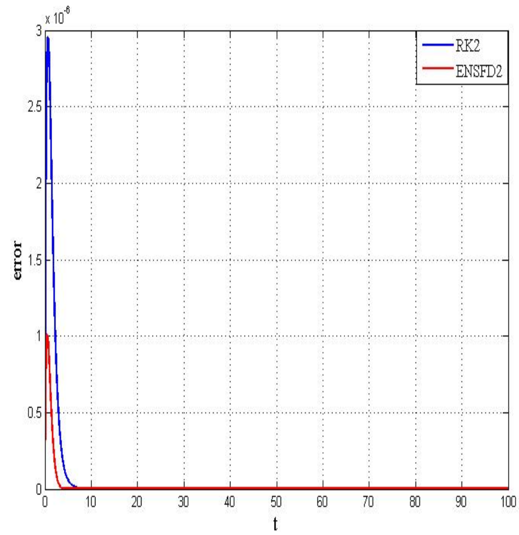

Figure 5. Errors of the ENSFD2 and RK2 schemes

IV. CONCLUSIONS AND DISCUSSIONS

As the main conclusion of this study, we have proposed and analyzed high-order numerical schemes that preserve the dynamical properties of a mathematical model describing the spread of computer viruses on the Internet. By combining the first-order NSFD scheme (2) with Richardson's extrapolation method, we have obtained extrapolated numerical schemes that not only preserve the dynamical properties of the model under consideration but also provide higher-accuracy numerical approximations. Finally, a set of numerical examples has been conducted to illustrate and support the theoretical findings and to show the advantages of the proposed numerical schemes.

The results and approach presented in this work will be useful in future works to construct high-order numerical schemes for mathematical models describing the spread of computer viruses and malware in particular and mathematical models arising in science and technology in general.

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